$$2 - \alpha = d\nu \,, \tag{7.2.18}$$

which also contains the spatial dimension d. According to equations (7.2.6), (7.2.9), (7.2.15), and (7.2.18), all the critical exponents are determined by two independent ones.

For the *two-dimensional Ising model* one finds the exponents of the correlation function, $\nu = 1$ and $\eta = 1/4$, from the exponents quoted following Eq. (6.5.31d) and the scaling relations (7.2.15) and (7.2.18).

7.3 The Renormalization Group

7.3.1 Introductory Remarks

The term 'renormalization' of a theory refers to a certain reparametrization with the goal of making the renormalized theory more easily dealt with than the original version. Historically, renormalization was developed by Stückelberg and Feynman in order to remove the divergences from quantum-field theories such as quantum electrodynamics. Instead of the bare parameters (masses, coupling constants), the Lagrange function is expressed in terms of physical masses and coupling coefficients, so that ultraviolet divergences due to virtual transitions occur only within the connection between the bare and the physical quantities, leaving the renormalized theory finite. The renormalization procedure is not unique; the renormalized quantities can for example depend upon a cutoff length scale, up to which certain virtual processes are taken into account. *Renormalization group theory* studies the dependence on this length scale, which is also called the "flow parameter". The name "renormalization group" comes from the fact that two consecutive renormalization group transformations lead to a third such transformation.

In the field of critical phenomena, where one must explain the observed behavior at large distances (or in Fourier space at small wavenumbers), it is reasonable to carry out the renormalization procedure by a suitable elimination of the short-wavelength fluctuations. A partial evaluation of the partition function in this manner is easier to carry out than the calculation of the complete partition function, and can be done using approximation methods. As a result of the elimination step, the remaining degrees of freedom are subject to modified, *effective* interactions.

Quite generally, one can expect the following advantages from such a renormalization group transformation:

(i) The new coupling constants could be smaller. By repeated applications of the renormalization procedure, one could thus finally obtain a practically free theory, without interactions.
(ii) The successively iterated coupling coefficients, also called "parameter flow", could have a *fixed point*, at which the system no longer changes

under additional renormalization group transformations. Since the elimination of degrees of freedom is accompanied by a change of the underlying lattice spacing, or length scale, one can anticipate that the fixed points are under certain circumstances related to critical points. Furthermore, it can be hoped that the flow in the vicinity of these fixed points can yield information about the universal physical quantities in the neighborhood of the critical points.

The scenario described under (i) will in fact be found for the one-dimensional Ising model, and that described under (ii) for the two-dimensional Ising model.

The renormalization group method brings to bear the scale invariance in the neighborhood of a critical point. In the case of so called *real-space transformations* (in contrast to transformation in Fourier space), one eliminates certain degrees of freedom which are defined on a lattice, and thus carries out a partial trace operation on the partition function. The lattice constant of the resulting system is then readjusted and the internal variables are renormalized in such a manner that the new Hamiltonian corresponds to the original one in its form. By comparison, one defines effective, scale-independent coupling constants, whose flow behavior is then investigated. We first study the one-dimensional Ising model and then the two-dimensional. Finally, the general structure of such transformations will be discussed with the derivation of scaling laws. A brief schematic treatment of continuous field-theoretical formulations will be undertaken following the Ginzburg–Landau theory.

7.3.2 The One-Dimensional Ising Model, Decimation Transformation

We will first illustrate the renormalization group method using the one-dimensional Ising model, with the ferromagnetic exchange constant J in zero applied field, as an example. The Hamiltonian is

$$\mathcal{H} = -J \sum_l \sigma_l \sigma_{l+1} \ , \tag{7.3.1}$$

where l runs over all the sites in the one-dimensional chain; see Fig. 7.4. We introduce the abbreviation $K = J/kT$ into the partition function for N spins with periodic boundary conditions $\sigma_{N+1} = \sigma_1$,

$$Z_N = \mathrm{Tr}\, \mathrm{e}^{-\mathcal{H}/kT} = \sum_{\{\sigma_l = \pm 1\}} \mathrm{e}^{K \sum_l \sigma_l \sigma_{l+1}} \ . \tag{7.3.2}$$

The decimation procedure consists in partially evaluating the partition function, by carrying out the sum over every second spin in the first step. In Fig. 7.4, the lattice sites for which the trace is taken are marked with a cross.

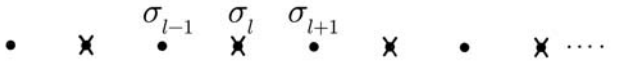

Fig. 7.4. An Ising chain; the trace is carried out over all the lattice points which are marked with a cross. The result is a lattice with its lattice constant doubled

A typical term in the partition function is then

$$\sum_{\sigma_l=\pm 1} \mathrm{e}^{K\sigma_l(\sigma_{l-1}+\sigma_{l+1})} = 2\cosh K(\sigma_{l-1}+\sigma_{l+1}) = \mathrm{e}^{2g+K'\sigma_{l-1}\sigma_{l+1}} \ , \quad (7.3.3)$$

with coefficients g and K' which are still to be determined. Here, we have taken the sum over $\sigma_l = \pm 1$ after the first equals sign. Since $\cosh K(\sigma_{l-1} + \sigma_{l+1})$ depends only on whether σ_{l-1} and σ_{l+1} are parallel or antiparallel, the result can in any case be brought into the form given after the second equals sign. The coefficients g and K' can be determined either by expansion of the exponential function or, still more simply, by comparing the two expressions for the possible orientations. If $\sigma_{l-1} = -\sigma_{l+1}$, we find

$$2 = \mathrm{e}^{2g-K'} \ , \quad (7.3.4a)$$

and if $\sigma_{l-1} = \sigma_{l+1}$, the result is

$$2\cosh 2K = \mathrm{e}^{2g+K'} \ . \quad (7.3.4b)$$

From the product of (7.3.4a) and (7.3.4b) we obtain $4\cosh 2K = \mathrm{e}^{4g}$, and from the quotient, $\cosh 2K = \mathrm{e}^{2K'}$; thus the recursion relations are:

$$K' = \frac{1}{2}\log\cosh 2K \quad (7.3.5a)$$

$$g = \frac{1}{2}\big(\log 2 + K'\big) \ . \quad (7.3.5b)$$

Repeating this decimation procedure a total of k times, we obtain from (7.3.5a,b) for the kth step the following recursion relation:

$$K^{(k)} = \frac{1}{2}\log\Big(\cosh 2K^{(k-1)}\Big) \quad (7.3.6a)$$

$$g(K^{(k)}) = \frac{1}{2}\log 2 + \frac{1}{2}K^{(k)} \ . \quad (7.3.6b)$$

The decimation produces another Ising model with an interaction between nearest neighbors having a coupling constant $K^{(k)}$. Furthermore, a spin-independent contribution $g(K^{(k)})$ to the energy is generated; in the kth step, it is given by (7.3.6b).

In a transformation of this type, it is expedient to determine the *fixed points* which in the present context will prove to be physically relevant. Fixed

points are those points K^* which are invariant with respect to the transformation, i.e. here $K^* = \frac{1}{2}\log(\cosh 2K^*)$. This equation has two solutions,

$$K^* = 0 \quad (T = \infty) \quad \text{and} \quad K^* = \infty \quad (T = 0)\,. \tag{7.3.7}$$

The recursion relation (7.3.6a) is plotted in Fig. 7.5. Starting with the initial value K_0, one obtains $K'(K_0)$, and by a reflection in the line $K' = K$, $K'(K'(K_0))$, and so forth. One can see that the coupling constant decreases continually; the system moves towards the fixed point $K^* = 0$, i.e. a non-interacting system. Therefore, for a finite K_0, we never arrive at an ordered state: there is no phase transition. Only for $K = \infty$, i.e. for a finite exchange interaction J and $T = 0$, do the spins order.

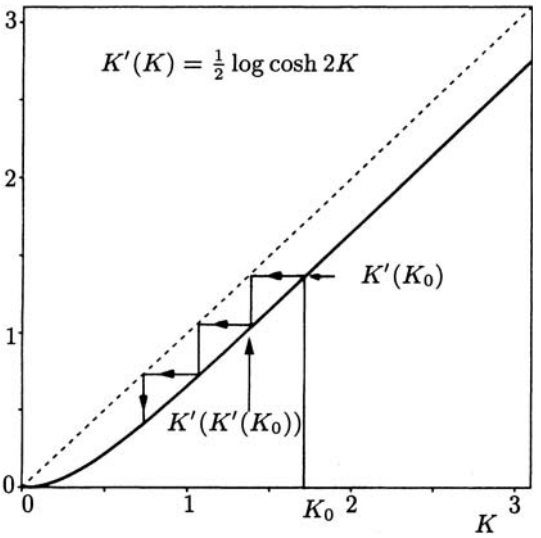

Fig. 7.5. The recursion relation for the one-dimensional Ising model with interactions between nearest neighbors (heavy solid curve), the line $K' = K$ (dashed), and the iteration steps (thin lines with arrows)

Making use of this renormalization group (RG) transformation, we can calculate the partition function and the free energy. The partition function for all together N spins with the coupling constant K, using (7.3.3), is

$$Z_N(K) = \mathrm{e}^{Ng(K')} Z_{\frac{N}{2}}(K') = \mathrm{e}^{Ng(K')+\frac{N}{2}g(K'')} Z_{\frac{N}{2^2}}(K'')\,, \tag{7.3.8}$$

and, after the nth step,

$$Z_N(K) = \exp\left[N\sum_{k=1}^{n}\frac{1}{2^{k-1}}g\big(K^{(k)}\big) + \log Z_{\frac{N}{2^n}}\big(K^{(n)}\big)\right]. \tag{7.3.9}$$

The reduced free energy per lattice site and kT is defined by

$$\tilde{f} = -\frac{1}{N}\log Z_N(K)\,. \tag{7.3.10}$$

As we have seen, the interactions become weaker as a result of the renormalization group transformation, which gives rise to the following possible application: after several steps the interactions have become so weak that perturbation-theory methods can be used, or the interaction can be altogether neglected. Setting $K^{(n)} \approx 0$, from (7.3.9) we obtain the approximation:

$$\tilde{f}^{(n)}(K) = -\sum_{k=1}^{n} \frac{1}{2^{k-1}} g\big(K^{(k)}\big) - \frac{1}{2^n} \log 2 \,, \tag{7.3.11}$$

since the free energy per spin of a field-free spin-1/2 system without interactions is $-\log 2$. Fig. 7.6 shows $\tilde{f}^{(n)}(K)$ for $n = 1$ to 5. We can see how quickly this approximate solution approaches the exact reduced free energy $\tilde{f}(K) = -\log(2\cosh K)$. The one-dimensional Ising model can be exactly solved by elementary methods (see problem 6.9), as well as by using the transfer matrix method, cf. Appendix F.

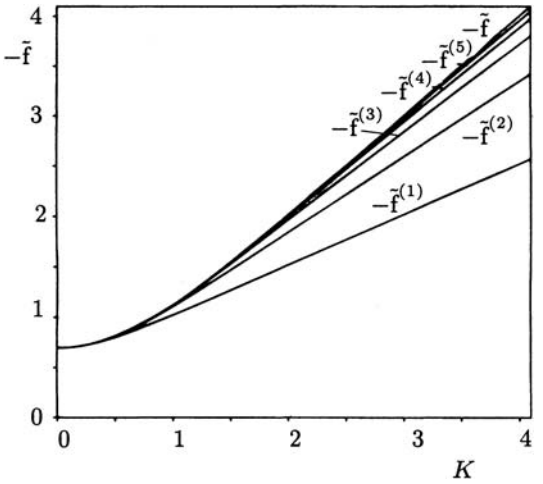

Fig. 7.6. The reduced free energy of the one-dimensional Ising model. $\tilde{f}$ is the exact free energy, $\tilde{f}^{(1)}, \tilde{f}^{(2)}, \ldots$ are the approximations (7.3.11)

7.3.3 The Two-Dimensional Ising Model

The application of the decimation procedure to the two-dimensional Ising model is still more interesting, since this model exhibits a phase transition at a finite temperature $T_c > 0$. We consider the square lattice rotated by 45° which is illustrated in Fig. 7.7, with a lattice constant of one.

The Hamiltonian multiplied by β, $H = \beta\mathcal{H}$, is

$$H = -\sum_{n.n.} K\sigma_i\sigma_j \,, \tag{7.3.12}$$

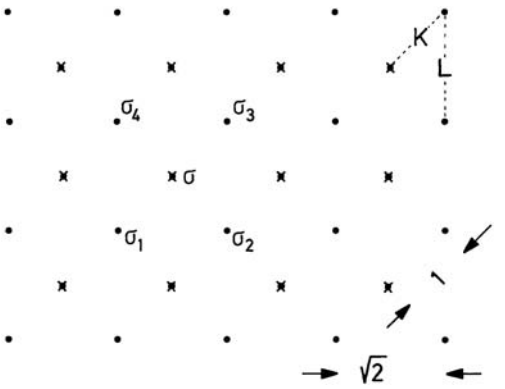

Fig. 7.7. A square spin lattice, rotated by 45°. The lattice sites are indicated by points. In the decimation transformation, the spins at the sites which are also marked by a cross are eliminated. K is the interaction between nearest neighbors and L is the interaction between next-nearest neighbors

where the sum runs over all pairs of nearest neighbors (n.n.) and $K = J/kT$. When in the partial evaluation of the partition function the trace is taken over the spins marked by crosses, we obtain a new square lattice of lattice constant $\sqrt{2}$. How do the coupling constants transform? We pick out one of the spins with a cross, σ, denote its neighbors as $\sigma_1, \sigma_2, \sigma_3$, and σ_4, and evaluate their contribution to the partition function:

$$\begin{aligned}\sum_{\sigma=\pm 1} \mathrm{e}^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\sigma} &= \mathrm{e}^{\log(2\cosh K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))} \\ &= \mathrm{e}^{A'+\frac{1}{2}K'(\sigma_1\sigma_2\ldots+\sigma_3\sigma_4)+L'(\sigma_1\sigma_3+\sigma_2\sigma_4)+M'\sigma_1\sigma_2\sigma_3\sigma_4} .\end{aligned} \tag{7.3.13}$$

This transformation (taking a partial trace) yields a modified interaction between nearest neighbors, K' (here, the elimination of two crossed spins contributes); in addition, new interactions between the next-nearest neighbors (such as σ_1 and σ_3) and a four-spin interaction are generated:

$$H' = \Big(A' + K' \sum_{n.N.} \sigma_i\sigma_j + L' \sum_{\text{"}u.n.N.} \sigma_i\sigma_j + \ldots\Big) . \tag{7.3.12'}$$

The coefficients A', K', L' and M' can readily be found from (7.3.13) as functions of K, by using ${\sigma_i}^2 = 1$, $i = 1, \ldots, 4$ (see problem 7.2):

$$A'(K) = \log 2 + \frac{1}{8}\big\{\log\cosh 4K + 4\log\cosh 2K\big\} , \tag{7.3.14}$$

$$K'(K) = \frac{1}{4}\log\cosh 4K , \quad L'(K) = \frac{1}{2}K'(K) \tag{7.3.13'}$$

$$M'(K) = \frac{1}{8}\big\{\log\cosh 4K - 4\log\cosh 2K\big\} .$$

Putting the critical value $K_c = J/kT_c = 0.4406$ (exact result[9]) into this relation as an estimate for the initial value K, we find $M' \ll L' \leq K'$. In

[9] The partition function of the Ising model on a square lattice without an external field was evaluated exactly by L. Onsager, Phys. Rev. **65**, 117 (1944), using the transfer matrix method (see Appendix F.).

the first elimination step, the original Ising model is transformed into one with three interactions; in the next step we must take these into account and obtain still more interactions, and so on. In a quantitatively usable calculation it will thus be necessary to determine the recursion relations for an extended number of coupling constants. Here, we wish only to determine the essential structure of such recursion relations and to simplify them sufficiently so that an analytic solution can be found. Therefore, we neglect the coupling constant M' and all the others which are generated by the elimination procedure, and restrict ourselves to K' and L' as well as their initial values K and L. This is suggested by the smallness of M' which we mentioned above.

We now require the recursion relation including the coupling constant L, which acts between σ_1 and σ_4, etc. Thus, expanding (7.3.13′) up to second order in K and taking note of the fact that an interaction L between next-nearest neighbors in the original Hamiltonian appears as a contribution to the interactions of the nearest neighbors in the primed Hamiltonian, we find the following recursion relations on elimination of the crossed spins (Fig. 7.7):

$$K' = 2K^2 + L \tag{7.3.15a}$$

$$L' = K^2 \,. \tag{7.3.15b}$$

These relations can be arrived at intuitively as follows: the spin σ mediates an interaction of the order of K times K, i.e. K^2 between σ_1 and σ_3, likewise the crossed spin just to the left of σ. This leads to $2K^2$ in K'. The interaction L between next-nearest neighbors in the original model makes a direct contribution to K'. Spin σ also mediates a diagonal interaction between σ_1 and σ_4, leading thus to the relation $L' = K^2$ in (7.3.15b).

However, it should be clear that in contrast to the one-dimensional case, new coupling constants are generated in every elimination step. One cannot expect that these recursion relations, which have been restricted as an approximation to a reduced parameter space (K, L), will yield quantitatively accurate results. They do contain all the typical features of this type of recursion relations.

In Fig. 7.8, we have shown the recursion relations (7.3.15a,b)[10]. Starting from values $(K, 0)$, the recursion relation is repeatedly applied, likewise for initial values $(0, L)$. The following picture emerges: for small initial values, the flux lines converge to $K = L = 0$, and for large initial values they converge to $K = L = \infty$. These two regions are separated by two lines, which meet at $K_c^* = \frac{1}{3}$ and $L_c^* = \frac{1}{9}$. Further on it will become clear that this fixed point is connected to the critical point.

We now want to investigate analytically the more important properties of the flow diagram which follows from the recursion relations (7.3.15a,b). As a

[10] For clarity we have drawn in only every other iteration step in Fig. 7.8. We will return to this point at the end of this section, after investigating the analytic behavior of the recursion relation.

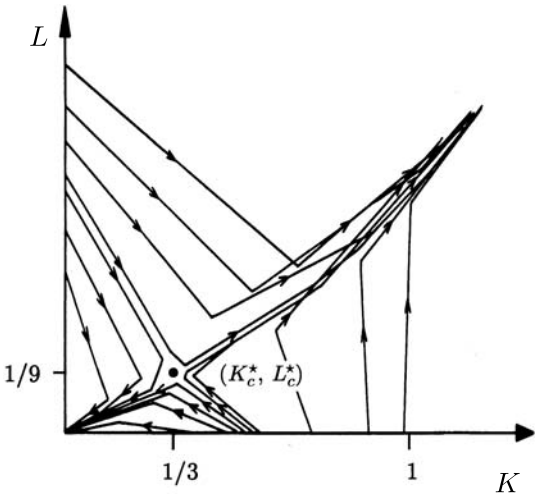

Fig. 7.8. A flow diagram of Eq. (7.3.15a,b) (only every other point is indicated.) Three fixed points can be recognized: $K^* = L^* = 0, K^* = L^* = \infty$ and $K_c^* = \frac{1}{3}, L_c^* = \frac{1}{9}$

first step, the *fixed points* must be determined from (7.3.15a,b), i.e. K^* and L^*, which obey $K^* = 2K^{*2} + L^*$ and $L^* = K^*$. These conditions give three fixed points

$$\text{(i)} \quad K^* = L^* = 0, \quad \text{(ii)} \quad K^* = L^* = \infty, \quad \text{and} \quad \text{(iii)} \quad K_c^* = \frac{1}{3},\ L_c^* = \frac{1}{9}\ . \tag{7.3.16}$$

The high-temperature fixed point (i) corresponds to a temperature $T = \infty$ (disordered phase), while the low-temperature fixed point (ii) corresponds to $T = 0$ (ordered low-temperature phase). The critical behavior can be related only to the non-trivial fixed point (iii), $(K_c^*, L_c^*) = (\frac{1}{3}, \frac{1}{9})$.

That the initial values of K and L which lead to the fixed point (K_c^*, L_c^*) represent critical points can be seen in the following manner: the RG transformation leads to a lattice with its lattice constant increased by a factor of $\sqrt{2}$. The correlation length of the transformed system ξ' is thus smaller by a factor of $\sqrt{2}$:

$$\xi' = \xi/\sqrt{2}\ . \tag{7.3.17}$$

However, at the fixed point, the coupling constants K_c^*, L_c^* are invariant, so that for ξ of the fixed point, we have $\xi' = \xi$, i.e. at the fixed point, it follows that $\xi = \xi/\sqrt{2}$, thus

$$\xi = \begin{cases} \infty & \text{or} \\ 0 & . \end{cases} \tag{7.3.18}$$

The value 0 corresponds to the high-temperature and to the low-temperature fixed points. At finite K^*, L^*, ξ cannot be zero, but only ∞. Calculating

back through the transformation shows that the correlation length at each point along the critical trajectory which leads to the fixed point is infinite. Therefore, all the points of the "critical trajectory", i.e. the trajectory leading to the fixed point, are critical points of Ising models with nearest-neighbor and next-nearest-neighbor interactions.

In order to determine the critical behavior, we examine the behavior of the coupling constants in the vicinity of the "non-trivial" fixed point; to this end, we linearize the transformation equations (7.3.15a,b) around (K_c^*, L_c^*) in the lth step:

$$\delta K_l = K_l - K_c^* \quad , \quad \delta L_l = L_l - L_c^* \; . \tag{7.3.19}$$

We thereby obtain the following linear recursion relation:

$$\begin{pmatrix} \delta K_l \\ \\ \delta L_l \end{pmatrix} = \begin{pmatrix} 4K_c^* & 1 \\ \\ 2K_c^* & 0 \end{pmatrix} \begin{pmatrix} \delta K_{l-1} \\ \\ \delta L_{l-1} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 1 \\ \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} \delta K_{l-1} \\ \\ \delta L_{l-1} \end{pmatrix} \; . \tag{7.3.20}$$

The eigenvalues of the transformation matrix can be determined from $\lambda^2 - \frac{4}{3}\lambda - \frac{2}{3} = 0$, i.e.

$$\lambda_{1,2} = \frac{1}{3}(2 \pm \sqrt{10}) = \begin{cases} 1.7208 \\ -0.3874 \; . \end{cases} \tag{7.3.21a}$$

The associated eigenvectors can be obtained from $\left(4 - (2 \pm \sqrt{10})\right) \delta K + 3\delta L = 0$, i.e.

$$\begin{aligned} &\delta L = \pm \frac{\sqrt{10} - 2}{3} \, \delta K \quad \text{and thus} \\ &\mathbf{e}_1 = \left(1, \frac{\sqrt{10} - 2}{3}\right) \quad \text{and} \quad \mathbf{e}_2 = \left(1, -\frac{\sqrt{10} + 2}{3}\right) \end{aligned} \tag{7.3.21b}$$

with the scalar product $\mathbf{e}_1 \cdot \mathbf{e}_2 = \frac{1}{3}$.

We now start from an Ising model with coupling constants K_0 and L_0 (including the division by kT). We first expand the deviations of the initial coupling constants K_0 and L_0 from the fixed point in the basis of the eigenvectors (7.3.21):

$$\begin{pmatrix} K_0 \\ L_0 \end{pmatrix} = \begin{pmatrix} K_c^* \\ L_c^* \end{pmatrix} + c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 \; , \tag{7.3.22}$$

with expansion coefficients c_1 and c_2. The decimation procedure is repeated several times; after l transformation steps, we obtain the coupling constants K_l and L_l :

$$\begin{pmatrix} K_l \\ L_l \end{pmatrix} = \begin{pmatrix} K_c^* \\ L_c^* \end{pmatrix} + \lambda_1^l c_1 \mathbf{e}_1 + \lambda_2^l c_2 \mathbf{e}_2 \; . \tag{7.3.23}$$

If the Hamiltonian H differs from H^* only by an increment in the direction $\mathbf{e}_2$, the successive application of the renormalization group transformation leads to the fixed point, since $|\lambda_2| < 1$ (see Fig. 7.9).

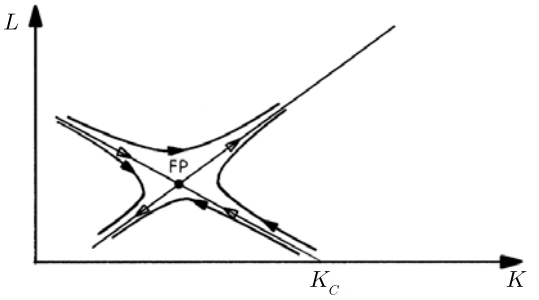

Fig. 7.9. Flow diagram based on the recursion relation (7.3.22), which is linearized around the nontrivial fixed point (FP)

Let us now consider the original nearest-neighbor Ising model with the coupling constant $K_0 \equiv \frac{J}{kT}$ and with $L_0 = 0$, and first determine the critical value K_c; this is the value of K_0 which leads to the fixed point. The condition for K_c, from the above considerations, is given by

$$\begin{pmatrix} K_c \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{9} \end{pmatrix} + 0 \cdot \mathbf{e}_1 + c_2 \begin{pmatrix} 1 \\ -\frac{\sqrt{10}+2}{3} \end{pmatrix} . \tag{7.3.24}$$

These two linear equations have the solution

$$c_2 = \frac{1}{3(\sqrt{10}+2)}, \quad \text{and therefore} \quad K_c = \frac{1}{3} + \frac{1}{3(\sqrt{10}+2)} = 0.3979 . \tag{7.3.25}$$

For $K_0 = K_c$, the linearized RG transformation leads to the fixed point, i.e. this is the critical point of the nearest-neighbor Ising model, $K_c = \frac{J}{kT_c}$. From the nonlinear recursion relation (7.3.15a,b), we find for the critical point the slighty smaller value $K_c^{n.l.} = 0.3921$. Both values differ from Onsager's exact solution, which gives $K_c = 0.4406$, but they are much closer than the value from molecular field theory, $K_c = 0.25$.

For $K_0 = K_c$, only $c_2 \neq 0$, and the transformation leads to the fixed point. For $K_0 \neq K_c$, we also have $c_1 \propto (K_0 - K_c) = -\frac{J}{kT_c^2}(T - T_c) \cdots \neq 0$. This increases with each application of the RG transformation, and thus leads away from the fixed point (K_c^*, L_c^*) (Fig. 7.9), so that the flow runs either to the low-temperature fixed point (for $T < T_c$) or to the high-temperature fixed point (for $T > T_c$).

Now we may determine the critical exponent ν for the correlation length, beginning with the recursion relation

$$(K - K_c)' = \lambda_1 (K - K_c) \tag{7.3.26}$$

and writing λ_1 as a power of the new length scale

$$\lambda_1 = (\sqrt{2})^{y_1} . \tag{7.3.27}$$

For the exponent y_1 defined here, we find the value

$$y_1 = 2\frac{\log \lambda_1}{\log 2} = 1.566 \,. \tag{7.3.28}$$

From $\xi' = \xi/\sqrt{2}$ (Eq. (7.3.17)), it follows that $(K'-K_c)^{-\nu} = (K-K_c)^{-\nu}/\sqrt{2}$, i.e.

$$(K' - K_c) = (\sqrt{2})^{\frac{1}{\nu}}(K - K_c) \,. \tag{7.3.29}$$

Comparing this with the first relation (7.3.26), we obtain

$$\nu = \frac{1}{y_1} = 0.638 \,. \tag{7.3.30}$$

This is, to be sure, quite a ways from 1, the known exact value of the two-dimensional Ising model, but nevertheless it is larger than 0.5, the value from the molecular-field approximation. A considerable improvement can be obtained by extending the recursion relation to several coupling coefficients.

Let us now consider the effect of a finite magnetic field h (including the factor β). The recursion relation can again be established intuitively. The field h acts directly on the remaining spins, as well as a (somewhat underestimated) additional field Kh which is due to the orienting action of the field on the eliminated neighboring spins, so that all together we have

$$h' = h + Kh \,. \tag{7.3.31}$$

The fixed point value of this recursion relation is $h^* = 0$. Linearization around the fixed point yields

$$h' = (1 + K^*)h = \frac{4}{3}h \,; \tag{7.3.32}$$

thus the associated eigenvalue is

$$\lambda_h = \frac{4}{3} \,. \tag{7.3.33}$$

$K_0 - K_c$ (or $T - T_c$) and h are called the *relevant "fields"*, since the eigenvalues λ_1 and λ_h are larger than 1, and they therefore increase as a result of the renormalization group transformation and lead away from the fixed point. In contrast, c_2 is an *"irrelevant field"*, since $|\lambda_2| < 1$, and therefore c_2 becomes increasingly smaller with repeated RG transformations. Here, "fields" refers to fields in the usual sense, but also to coupling constants in the Hamiltonian. The structure found here is typical of models which describe critical points, and remains the same even when one takes arbitrarily many coupling constants into account in the transformation: there are *two relevant fields* ($T - T_c$ and h, the conjugate field to the order parameter), and *all the other fields are irrelevant.*

We add a remark concerning the flow diagram 7.9. There, owing to the negative sign of λ_2, only every other point is shown. This corresponds to a twofold application of the transformation and an increase of the lattice constant by a factor of 2, as well as $\lambda_1 \to \lambda_1^2, \lambda_2 \to \lambda_2^2$. Then the second eigenvalue λ_2^2 is also positive, since otherwise the trajectory would move along an oscillatory path towards the fixed point.

7.3.4 Scaling Laws

Although the decimation procedure described in Sect. 7.3.3 with only a few parameters does not give quantitatively satisfactory results and is also unsuitable for the calculation of correlation functions, it does demonstrate the general structure of RG transformations, which we shall now use as a starting point for deriving the scaling laws.

A general RG transformation $\mathcal{R}$ maps the original Hamiltonian $\mathcal{H}$ onto a new one,

$$\mathcal{H}' = \mathcal{R}\mathcal{H} \; . \tag{7.3.34}$$

This transformation also implies the rescaling of all the lengths in the problem, and that $N' = Nb^{-d}$ holds for the number of degrees of freedom N in d dimensions (here, $b = \sqrt{2}$ for the decimation transformation of 7.3.1).

The fixed-point Hamiltonian is determined by

$$\mathcal{R}(\mathcal{H}^*) = \mathcal{H}^* \; . \tag{7.3.35}$$

For small deviations from the fixed-point Hamiltonian,

$$\mathcal{R}(\mathcal{H}^* + \delta\mathcal{H}) = \mathcal{H}^* + \mathcal{L}\,\delta\mathcal{H} \; ,$$

we can expand in terms of the deviation $\delta\mathcal{H}$. From the expansion, we obtain the linearized recursion relation

$$\mathcal{L}\delta\mathcal{H} = \delta\mathcal{H}' \; . \tag{7.3.36a}$$

The eigenoperators $\delta\mathcal{H}_1, \delta\mathcal{H}_2, \ldots$ of this linear transformation are determined by the eigenvalue equation

$$\mathcal{L}\delta\mathcal{H}_i = \lambda_i \delta\mathcal{H}_i \; . \tag{7.3.36b}$$

A given Hamiltonian $\mathcal{H}$, which differs only slightly from $\mathcal{H}^*$, can be represented by $\mathcal{H}^*$ and the deviations from it:

$$\mathcal{H} = \mathcal{H}^* + \tau\delta\mathcal{H}_\tau + h\delta\mathcal{H}_h + \sum_{i\geq 3} c_i \delta\mathcal{H}_i \; , \tag{7.3.37}$$

where $\delta\mathcal{H}_\tau$ and $\delta\mathcal{H}_h$ denote the two *relevant* perturbations with

$$|\lambda_\tau| = b^{y_\tau} > 1 \; , \; |\lambda_h| = b^{y_h} > 1 \; ; \tag{7.3.38}$$

they are related to the temperature variable $\tau = \frac{T-T_c}{T_c}$ and the external field h, while $|\lambda_j| = b^{y_j} < 1$ and thus $y_j < 0$ for $j \geq 3$ are connected with the *irrelevant* perturbations.[11] The coefficients τ, h, and c_j are called *scaling*

[11] Compare the discussion following Eq. (7.3.33). The (only) irrelevant field there is denoted by c_2. In the following, we assume that $\lambda_i \geq 0$.

fields. For the Ising model, $\delta\mathcal{H}_h = \sum_l \sigma_l$. Denoting the initial values of the fields by c_i, we find that the free energy transforms after l steps to

$$F_N(c_i) = F_{N/b^{dl}}(c_i \lambda_i^l) \; . \tag{7.3.39a}$$

For the free energy per spin,

$$f(c_i) = \frac{1}{N} F_N(c_i) \; , \tag{7.3.39b}$$

we then find in the linear approximation

$$f(\tau, h, c_3, \ldots) = b^{-dl} f\big(\tau b^{y_\tau l}, h b^{y_h l}, c_3 b^{y_3 l}, \ldots\big) \; . \tag{7.3.40}$$

Here, we have left off an additive term which has no influence on the following derivation of the scaling law; it is, however, important for the calculation of the free energy. The scaling parameter l can now be chosen in such a way that $|\tau| b^{y_\tau l} = 1$, which makes the first argument of f equal to ± 1. Then we find

$$f(\tau, h, c_3, \ldots) = |\tau|^{d/y_\tau} \hat{f}_\pm\big(h|\tau|^{-y_h/y_\tau}, c_3 |\tau|^{|y_3|/y_\tau}, \ldots\big) \; , \tag{7.3.40$'$}$$

where $\hat{f}_\pm(x, y, \ldots) = f(\pm 1, x, y, \ldots)$ and $y_\tau, y_h > 0, y_3, \ldots < 0$. Close to T_c, the dependence on the irrelevant fields $c_3, \ldots$ can be neglected, and Eq. (7.3.40$'$) then takes on precisely the scaling form (Eq. 7.2.7), with the conventional exponents

$$\beta\delta = y_h / y_\tau \tag{7.3.41a}$$

and

$$2 - \alpha = \frac{d}{y_\tau} \; . \tag{7.3.41b}$$

Taking the derivative with respect to h yields

$$\beta = \frac{d - y_h}{y_\tau} \quad \text{and} \quad \gamma = \frac{d - 2y_h}{y_\tau} \; . \tag{7.3.41c,d}$$

We have thus derived the scaling law, Eq. (7.2.7), within the RG theory for fixed points with just one relevant field, along with the applied magnetic field and the irrelevant operators. Furthermore, the dependence on the irrelevant fields $c_3, \ldots$ gives rise to corrections to the scaling laws, which must be taken into account for temperatures outside the asymptotic region.

In order to make the connection between y_τ and the exponent ν, we recall that l iterations reduce the correlation length to $\xi' = b^{-l}\xi$, which implies that $(\tau b^{y_\tau l})^{-\nu} = b^{-l} \tau^{-\nu}$ and, as a result,

$$\nu = \frac{1}{y_\tau} \tag{7.3.41e}$$

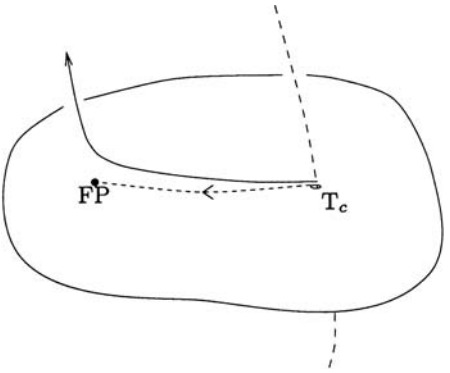

Fig. 7.10. The critical hypersurface. A trajectory within the critical hypersurface is shown as a dashed curve. The full curve is a trajectory near the critical hypersurface. The coupling coefficients of a particular physical system as a function of the temperature are indicated by the long-dashed curve

(cf. Eq. (7.3.30) for the two-dimensional Ising model). From the existence of a fixed-point Hamiltonian with two relevant operators, the scaling form of the free energy can be derived, and it is also possible to calculate the critical exponents. Even the form of the scaling functions $\hat{f}$ and $\hat{m}$ can be computed with perturbation-theoretical methods, since the arguments are finite. A similar procedure can be applied to the correlation function, Eq. (7.2.12b′). At this point it is important to renormalize the spin variable, $\sigma' = b^{\zeta}\sigma$, whereby it is found that setting the value

$$\zeta = (d - 2 + \eta)/2 \tag{7.3.41f}$$

guarantees the validity of (7.2.13) at the critical point.

We add a few remarks about the generic structure of the flow diagram in the vicinity of a critical fixed point (Fig. 7.10). In the multidimensional space of the coupling coefficients, there is a direction (the relevant direction) which leads away from the fixed point (we assume that $h = 0$). The other eigenvectors of the linearized RG transformation span the critical hypersurface. Further away from the fixed point, this hypersurface is no longer a plane, but instead is curved. The trajectories from each point on the critical hypersurface lead to the critical fixed point. When the initial point is close to but not precisely on the critical hypersurface, the trajectory at first runs parallel to the hypersurface until the relevant portion has become sufficiently large so that finally the trajectory leaves the neighborhood of the critical hypersurface and heads off to either the high-temperature or the low-temperature fixed point. For a given physical system (ferromagnet, liquid, ...), the parameters $\tau, c_3, \ldots$ depend on the temperature (the long-dashed curve in Fig. 7.10). The temperature at which this curve intersects the critical hypersurface is the transition temperature T_c.

From this discussion, the *universality properties* should be apparent. All systems which belong to a particular part of the parameter space, i.e. to the region of attraction of a given fixed point, are described by the same power laws in the vicinity of the critical hypersurface of the fixed point.